MASS TRANSFER BETWEEN A SPHERICAL PARTICLE AND A LIQUID FOR SMALL PECLET NUMBERS

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Analytic expressions are obtained in calculating the concentration field during nonstationary mass exchange between a spherical particle and a liquid flow.

The problem of convective diffusion from a single particle was investigated by Acrivos and Taylor [1]. More complicated problems, such as dependence of convective diffusion on coordinates [2], presence of absorption [3], and so on, were also investigated by these methods. Buevich and Perminov [3] considered nonstationary processes of heat- or mass-transfer from bodies of various shapes for the case of a streamline flow of these bodies around a potential flow for large Péclet numbers in the approximation of a diffusive boundary layer.

In the present paper we investigate the nonstationary mass transfer between a rare liquid or solid sphere and a viscous liquid flow, when the Rybchinskii—Hadamard equations are valid. It is also assumed that $\mathrm{Pe} \to 0$. Detailed analysis of the numerical solution of the equations of convective diffusion was earlier provided [4, 5] for similar problems.

We write the equations for the dimensionless concentrations ξ and η outside and inside the sphere, respectively:

$$\frac{\partial \xi}{\partial t} = \mathrm{Pe}\left(u_r^{(0)} \frac{\partial \xi}{\partial r} + \frac{u_\theta^{(0)}}{r} \frac{\partial \xi}{\partial \theta}\right) = \Delta \xi, \tag{1}$$

$$\gamma \frac{\partial \eta}{\partial t} = \gamma \mathrm{Pe}\left(u_r^{(i)} \frac{\partial \eta}{\partial r} + \frac{u_\theta^{(i)}}{r} \frac{\partial \eta}{\partial \theta}\right) = \Delta \eta. \tag{2}$$

The initial and boundary conditions imposed on the solution of (1)-(2) are given in the form

$$\xi = 0, \quad \eta = 1, \quad t = 0; \tag{3}$$

$$\xi \to 0, \quad r \to \infty, \quad \frac{\partial \eta}{\partial r} = 0, \quad r = 0; \tag{4}$$

$$\xi|_{r=1} = \eta|_{r=1}, \quad \gamma_1 \left.\frac{\partial \xi}{\partial r}\right|_{r=1} = \left.\frac{\partial \eta}{\partial r}\right|_{r=1}. \tag{5}$$

Here

$$\gamma = \frac{D_\xi}{D_\eta}; \quad t = \mathrm{Fo} = \frac{D_\xi}{a^2} t^*; \quad \mathrm{Pe} = \frac{aU}{D_\xi}; \quad \gamma_1 = \frac{\gamma}{\mathrm{H}}, \quad r = \frac{r^*}{a}, \tag{6}$$

and $u_r^{(o)}$, $u_\theta^{(o)}$, $u_r^{(i)}$, $u_\theta^{(i)}$ are the velocity components for the following current functions [6]:

$$\Psi^{(0)} = \sin^2\theta \left(-\frac{Br}{2} + Cr^2 + \frac{D}{r}\right), \tag{7}$$

$$\Psi^{(i)} = \sin^2\theta \left(\frac{Er^4}{10} + Gr^2\right). \tag{8}$$

Here and below the subscripts i or η are used for the internal part of the sphere, and the subscripts 0 or ξ for the space surrounding the sphere. The quantities appearing in Eqs. (7)-(8) acquire the values:

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$$C=\frac{1}{2},\quad B=\frac{1}{1+\sigma}\left(\frac{3}{2}+\sigma\right),\quad D=\frac{1}{4}\frac{1}{1+\sigma},$$
$$E=\frac{5}{2}\frac{\sigma}{1+\sigma},\quad G=-\frac{1}{4}\frac{\sigma}{1+\sigma},\quad \sigma=\frac{\mu_0}{\mu_i}. \tag{9}$$

The Henry number H, being a function of temperature and pressure, and being the ratio of the real material concentration at the inside boundary of the sphere to the concentration at the outer boundary, will be assumed to be a constant quantity.

Applying a Laplace transform to (1)-(5), we write the solution of the equations for ξ^* and η^* in form of asymptotic expansions in powers of Pe

$$\xi^*=\sum_{n=0}^{\infty}(\mathrm{Pe})^n\xi_n^*,\quad \eta=\sum_{n=0}^{\infty}(\mathrm{Pe})^n\eta_n^*. \tag{10}$$

For the zeroth approximation we obtain the Helmholtz equations, whose solutions are

$$\xi_0^*=\frac{(\sqrt{s\gamma}\,\mathrm{ch}\sqrt{s\gamma}-\mathrm{sh}\sqrt{s\gamma})\exp[-\sqrt{s}\,(r-1)]}{rs[\sqrt{s\gamma}\,\mathrm{ch}\sqrt{s\gamma}+(\gamma_1+\gamma_1\sqrt{s}-1)\,\mathrm{sh}\sqrt{s\gamma}]}, \tag{11}$$

$$\eta_0^*=\frac{1}{s}-\frac{\gamma_1(1+\sqrt{s})\,\mathrm{sh}(\sqrt{s\gamma}\,r)}{rs[\sqrt{s\gamma}\,\mathrm{ch}\sqrt{s\gamma}+(\gamma_1+\gamma_1\sqrt{s}-1)\,\mathrm{sh}\sqrt{s\gamma}]}, \tag{12}$$

where s is the Laplace variable.

In Eqs. (11)-(12) we replace the expressions for ξ_0^* and η_0^* by equivalent ones for small and large s, and, further applying the inverse Laplace transform, we obtain

$$\xi_0=\frac{\mathrm{H}}{3r}\left[\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)-\exp(r-1+t)\,\mathrm{erfc}\left(\sqrt{t}+\frac{k}{2\sqrt{t}}\right)\right], \tag{13}$$

$$\eta_0=1-\frac{1}{r}\left[\sqrt{\frac{t}{\pi}}\left(\exp\left[-\frac{k_1^2}{4t}\right]-\exp\frac{k_2^2}{4t}\right]\right)-\frac{1}{2}\left(k_1\,\mathrm{erfc}\frac{k_1}{2\sqrt{t}}-k_2\,\mathrm{erfc}\frac{k_2}{2\sqrt{t}}\right)\right] \tag{14}$$

for large t values, and

$$\xi_0=\frac{b}{\gamma_1-1}\frac{1}{r}\left[\left(\sqrt{\gamma}+\frac{1}{b_2}\right)\exp(b_2k+b_2^2t)\,\mathrm{erfc}\left(b_2\sqrt{t}+\frac{k}{2\sqrt{t}}\right)-\frac{1}{b_2}\,\mathrm{erfc}\frac{k}{2\sqrt{t}}\right], \tag{15}$$

$$\eta_0=1-\frac{\gamma_1}{\gamma_1-1}\frac{1}{r}\left[\mathrm{erfc}\frac{k_1}{2\sqrt{t}}+(b_2-1)\exp(b_2k_1+b_2^2t)\,\mathrm{erfc}\left(b_2\sqrt{t}+\frac{k_1}{2\sqrt{t}}\right)\right] \tag{16}$$

for small t.

For the case $\gamma=1$ and $H=1$ one can find the original transforms (11)-(12) exactly:

$$\xi_0=\frac{1}{r}\sqrt{\frac{t}{\pi}}\left[\exp\left(-\frac{k_0^2}{4t}\right)-\exp\left(-\frac{k^2}{4t}\right)\right]+\frac{1}{2}\left[\mathrm{erfc}\frac{k}{2\sqrt{t}}-\mathrm{erfc}\frac{k_0}{2\sqrt{t}}\right], \tag{17}$$

$$\eta_0=1-\frac{1}{2}\left[\mathrm{erfc}\frac{k_0}{2\sqrt{t}}+\mathrm{erfc}\left(-\frac{k}{2\sqrt{t}}\right)\right]-\frac{1}{r}\sqrt{\frac{t}{\pi}}\left[\exp\left(-\frac{k^2}{4t}\right)-\exp\left(-\frac{k_0^2}{4t}\right)\right], \tag{18}$$

where $k=r-1$; $k_0=r+1$; $k_1=\sqrt{\gamma}(1-r)$; $k_2=\sqrt{\gamma}(1+r)$; $b_2=(\gamma_1-1)(\gamma_1+\sqrt{\gamma})^{-1}$.

We seek solutions of the first approximation equations in the form

$$\xi_1^*=R_1^*(s,r)\cos\theta,\quad \eta_1^*=R_2^*(s,r)\cos\theta. \tag{19}$$

Substituting Eq. (19) in the corresponding equations and separating variables, we obtain the modified Bessel equations, whose solutions are found approximately. After the inverse transformations we have

$$R_1=-\frac{1}{18}\frac{r}{\sqrt{\pi t^3}}\exp\left(-\frac{r^2}{4t}\right)\left[\frac{\gamma a_1}{r^2(2\gamma_1+1)}+3C-\frac{3B}{2r}-\frac{3D}{2r^3}\right], \tag{20}$$

$$R_2=-\frac{1}{18}\frac{r\sqrt{\gamma}}{\sqrt{\pi t^3}}\exp\left(-\frac{r^2}{4t}\right)\left[\frac{\gamma a_1 r}{2\gamma_1+1}-\frac{3}{5}\gamma^2\left(\frac{1}{14}Er^5+Gr^3\right)\right] \tag{21}$$

for $t \to \infty$ and

$$R_1 = -\frac{1}{r}\left[\frac{a_3}{\sqrt{\gamma}(\gamma_1+\sqrt{\gamma})^2} + \frac{\sqrt{\gamma}}{\gamma_1+\sqrt{\gamma}}\left(2Cr - \ln r - \frac{D}{r^2}\right)\right] \operatorname{erfc}\frac{r-1}{2\sqrt{t}}, \tag{22}$$

$$R_2 = -\frac{1}{r}\left[\frac{a_4}{\sqrt{\gamma}(\gamma_1+\sqrt{\gamma})^2} + \frac{\gamma\gamma_1}{\gamma_1+\sqrt{\gamma}}\left(\frac{2}{15}Er^3 + 2Gr\right)\right] \operatorname{erfc}\frac{\sqrt{\gamma}(1-r)}{2\sqrt{t}} \tag{23}$$

for $t \to 0$, where

$$a_1 = \frac{3}{14}E + \frac{9}{5}G + \frac{3}{2}B + \frac{9}{2}D - \gamma\left(\frac{3}{70}E + \frac{3}{5}G\right) + \frac{\mathrm{H}}{\gamma}\left(-3C + \frac{3}{2}B + \frac{3}{2}D\right);$$

$$a_2 = \frac{3}{14}E + \frac{9}{5}G + 6C - \frac{3}{2}D + \frac{2\gamma^2}{H}\left(\frac{3}{70}E + \frac{3}{5}G\right);$$

$$a_3 = \frac{\gamma^2}{\mathrm{H}}\left[(1-\gamma)\left(\frac{2}{15}E + 2G\right) - \left(1 + \frac{H}{\sqrt{\gamma}}\right)(2C - D)\right];$$

$$a_4 = (1+\sqrt{\gamma})\frac{\gamma^3}{\mathrm{H}}\left(\frac{2}{15}E + 2G\right).$$

Thus, the solutions of Eqs. (1)-(2) can be written in first approximation in the form

$$\xi = \xi_0 + \mathrm{Pe}\, R_1(t, r)\cos\theta + O(\mathrm{Pe}^2), \tag{24}$$

$$\eta = \eta_0 + \mathrm{Pe}\, R_2(t, r)\cos\theta + O(\mathrm{Pe}^2). \tag{25}$$

For calculations of local Sherwood numbers we use the following equations:

$$\mathrm{Sh}_{\xi t} = -\frac{1}{\mathrm{H}}\left(\frac{\partial\xi}{\partial r}\right)\Big|_{r=1}, \quad \mathrm{Sh}_{\eta t} = -\left(\frac{\partial\eta}{\partial r}\right)\Big|_{r=1}. \tag{26}$$

Integrating $\mathrm{Sh}_{\xi t}$ and $\mathrm{Sh}_{\eta t}$ over the particle surface, it is easily seen that the mean Sherwood numbers are independent of the second terms of expansions (24)-(25), and therefore satisfy only the zeroth approximation

$$\overline{\mathrm{Sh}}_{\xi t} = \frac{2}{\mathrm{H}+\sqrt{\gamma}}\frac{1}{\sqrt{\pi t}} - \frac{2}{\mathrm{H}(\gamma_1-1)}[1 + (b_2\sqrt{\gamma}+1)(b_2-1)$$

$$\times \exp(b_2^2 t)\operatorname{erfc}(b_2\sqrt{t})]\frac{2\gamma_1\sqrt{\gamma}}{\mathrm{H}(\sqrt{\gamma}-\gamma_1)^2}\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{\gamma}{t}\right) + O\left[\sqrt{t}\exp\left(-\frac{\gamma}{t}\right)\right], \tag{27}$$

$$\overline{\mathrm{Sh}}_{\eta t} = \frac{2\gamma}{\mathrm{H}+\sqrt{\gamma}}\frac{1}{\sqrt{\pi t}} - \frac{2\gamma_1}{\gamma_1 - 1}\left[1 + (b_2-1)(b_2\sqrt{\gamma}+1)\right.$$

$$\left.\times \exp(b_2^2 t)\operatorname{erfc}(b_2\sqrt{t})\right] + \frac{2\gamma_1\sqrt{\gamma}}{\sqrt{\gamma}+\gamma_1}\frac{1}{\sqrt{\pi t}}\left[\exp\left(-\frac{\gamma}{t}\right) + \frac{\sqrt{\gamma}-\gamma_1}{\sqrt{\gamma}+\gamma_1}\right.$$

$$\left.\times\left[\exp\left(-\frac{\gamma}{4t}\right)\right]\right] + O\left[\sqrt{t}\exp\left(-\frac{\gamma}{4t}\right)\right]. \tag{28}$$

Expressions (27)-(28) are correct for small Fourier numbers, while their first terms coincide with the corresponding quantities derived in [4, 5].

When $\gamma = \mathrm{H} = 1$ the following equality is valid for any t

$$\overline{\mathrm{Sh}}_{\xi t} = \overline{\mathrm{Sh}}_{\eta t} = -2\left(\sqrt{\frac{t}{\pi}}\left[1 - \exp\left(-\frac{1}{t}\right)\right] - \frac{1}{2\sqrt{\pi t}}\left[1 - \exp\left(-\frac{1}{t}\right)\right]\right). \tag{29}$$

It is also interesting to determine the effective values of the Sherwood numbers after some time intervals. Figure 1 illustrates the dependence of Sh_ξ on the Fourier number. For

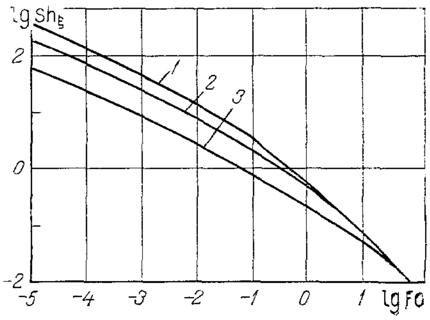

Fig. 1. The Sherwood number Sh_ξ as a function of the Fourier number Fo: 1) $\gamma = H = 1$; 2) $\gamma = 10$; $H = 0.1$; 3) $\gamma = 0.1$; $H = 10$.

sufficiently large Fo we have $Sh_\xi = 0.667\ (Fo)^{-1}$, which is verified by theoretical calculations [4].

NOTATION

Here a, radius of the spherical particle; C, B, D, E, G, σ, quantities defined in (9); D_ξ, D_η, diffusion coefficients; H, Henry number; Pe, Peclet number; r*, radial coordinate; r, dimensionless radial coordinate; Fo, Fourier number; t*, time, $Sh_{\xi t}$, $Sh_{\eta t}$, local Sherwood numbers, the bar denotes averaging over surface particles; Sh_ξ, Sh_η, effective Sherwood numbers; U, flow velocity; $u_r^{(o)}$, $u_\theta^{(o)}$, $u_r^{(i)}$, $u_\theta^{(i)}$, velocity components; γ, γ_1, quantities defined in (6); θ, angular coordinate; μ_o, μ_i, viscosities inside and outside the sphere; ξ, η, dimensionless concentrations; and $\psi^{(i)}$, $\psi^{(o)}$, stream flow.

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